A Further Note on Båth's Law

David Vere-Jones[1], *Junko Murakami*[2] *and Annemarie Christophersen*[2]

[1]Victoria University and Statistical Research Associates, Wellington, New Zealand
[2]Victoria University, Wellington, New Zealand
Corresponding Author: David Vere-Jones, dvj@mcs.vuw.ac.nz

Abstract: A simple model for the number and magnitudes of events in an aftershock sequences is explored. Two components are distinguished: the growth of observed aftershock numbers as the magnitude threshold is lowered, and the growth of aftershock numbers as the magnitude of the initiating event is increased. For a self-similar regime, the two growth rates should be the same. The distribution of the difference between the magnitude of the initiating event, and that of the largest aftershock in the sequence, is explored, and shown to provide a close approximation to Båth's law as well as a simple form for the probability of an initiating event to be followed by an aftershock of larger magnitude.

Introduction

The question of whether the so-called Båth's law has a physical basis, or is simply the consequence of basic statistical features of aftershock sequences, has remained controversial. In an early study, Vere-Jones (1969) pointed out that even without any physical basis, the distribution of the difference, say Δ, between the magnitudes of the largest and second largest events in an arbitrary sequence of events should be independent of the number of events in the sequence, so suggesting some form of statistical origin for the law. Recent papers by Lombardi (2002) and Console et al. (2003) explored this possibility in much greater depth, and concluded it was still possible that some additional, physically-based, component was needed to explain the observational results. Even more recently, Sornette and coworkers (see eg Helmsberger and Sornette Sornette (2003), Saichev et al. (2004), Zhuang (2005)) have made detailed studies of the generation of aftershocks in the ETAS model and established conditions under wich Båth's law appears, while Shcherbakov et al. (2004, 2005) have proposed modified versions of the law from a more physical viewpoint.

The present note explores what seems to us the simplest model for the numbers and magnitudes of the events in an aftershock sequence, and was prompted by the desire to interpret the modelling sugestions of some of the earlier papers in a simpler general setting.

Model description

We suppose first that aftershock magnitudes are independently and identically distributed above some threshold M_0 with common distribution

$$1 - F(M) = \Pr\{Mag > M\} = e^{-\beta(M-M_0)} = 10^{-b(M-M_0)}\ ; \quad \beta = 2.30b. \qquad (1)$$

(Note: we shall use logarithms to base e in the remainder of the discussion, and hence β rather than the more familiar b-value).

Denoting the magnitude of the initiating event by M^*, we suppose next that the expected number $\mu(M^*)$ of aftershocks with magnitudes over the threshold M_0 satisfies a scaling relation of the form

$$\mu(M^*) = Ae^{\alpha(M^*-M_0)}\ ; \quad \log[\mu(M^*)] = \log A + \alpha(M^* - M_0). \qquad (2)$$

Here M_0 plays the role of a scaling origin, and is not necessarily related to any catalogue completeness threshold, while A represents the expected number of aftershock events with $M \geq M_0$ if the initiating event itself has magnitude M_0.

From these assumptions alone it follows that, if we now take some higher threshold M_c determined by considerations of catalogue completeness or other reasons, the expected number of events above M_c, given an initiating event of size M^*, is given by

$$\mu(M^* \mid M_c) = p\mu(M^*) = e^{-\beta(M_c-M_0)}A\, e^{\alpha(M^*-M_0)}, \qquad (3)$$

since each of the initial events has the same probability $p = e^{-\beta(M_c-M_0)}$ of reaching the threshold M_c. Setting $\delta = \alpha - \beta$, we rewrite this in the form

$$\mu(M^* \mid M_c) = Ae^{\beta(M^*-M_c)}e^{(\alpha-\beta)(M^*-M_0)} = Ae^{\beta(M^*-M_c)}\, e^{\delta(M^*-M_0)}. \qquad (4)$$

which shows that the expected number depends on two separate features: the parameter β (or b) of the G-R law; and the discrepancy $\delta = \alpha - \beta$ of the two growth parameters.

If the threshold M_c is chosen relative to M^* itself, so that $M_c = M^* - D$, the equation becomes

$$\mu(M^* \mid M^* - D) = Ae^{\beta D}\, e^{\delta(M^*-M_0)}. \qquad (5)$$

From this form it is clear that the constraint $\alpha = \beta$ is essential if the aftershock process is to be self-similar in regard to the magnitude of the initiating event. It also provides a basis for treating an initial event as the foreshock of a larger event on a minimum of assumpions concerning the earthquake model, for setting $D = 0$ we obtain

$$\mu(M^* \mid M^*) = Ae^{\delta(M^*-M_0)} \qquad (6)$$

which provides an approximation to the probability that an initiating event of magnitude M^* generates an aftershock larger than itself.

Distribution of the discrepancy between M^* and the largest aftershock

More detailed results can be obtained by inserting specific assumptions for the distribution of the number of aftershocks. Let $G_N(z) = E(z^N)$ be the probability generating function (p.g.f.) of the number $N \equiv N(M^*)$ of aftershock events with $M > M_0$, initiated by an event of magnitude M^*. Conditional on $N = n$, and assuming independent magnitudes following the G-R law, the distribution of the largest of these events is given by

$$Pr(M_{max} \leq M \mid N(M^*) = n) = \left(1 - e^{-\beta(M-M_0)}\right)^n, \tag{7}$$

with the convention that $M_{max} = M_0$ if $N = 0$. Taking expectations over n then leads to the expression

$$F_{max}(M) = G_N\left[1 - e^{-\beta(M-M_0)}\right]. \tag{8}$$

for the overall distribution of the maximum. Specific examples can now be studied by selecting a particular form for the distribution of $N(M^*)$. One of the easiest to handle, which we use here for illustration, is to suppose N has a Poisson distribution with parameter $\mu(M^*)$, in which case (7) becomes

$$Pr(M_{max} \leq M \mid M^*) = e^{-\mu(M^*)\, e^{-\beta M}} \tag{9}$$

Substituting for $\mu(M^*)$, we obtain for the distribution of $\Delta = M^* - M_{max}$,

$$1 - F_\Delta(x) = Pr(\Delta \geq x) = Pr(M_{max} \leq M^* - x) = e^{-Ae^{\delta\,(M^*-M_0)}\, e^{\beta x}}.$$

Note that the distribution is limited to the region $\Delta \leq M^*$, with negative values corresponding to situations where the largest aftershock has a magnitude above that of the initiating event, and we need to take note of the convention $1 - F_\Delta(M^*) = \Pr\{M_{max} = 0)\} = \Pr\{N(M^*) = 0\}$.

The density $f_\Delta(x)$ attains its maximum at $x^* = -\frac{1}{\beta}(\log A + \delta(M^* - M_0))$; this mode is around 1 (i.e., the Båth's law value) when $\log A = -\delta(M^* - M_0) - \beta$. When $\delta = 0$ and $\beta = 2.3$, this gives $A = e^{-2.3} \approx 10\%$ for the chance of an initiating event being followed by a larger aftershock. Graphs of the density $f_\Delta(x)$ are shown in Figure 1 for a few choices of the parameters A and δ and M_0.

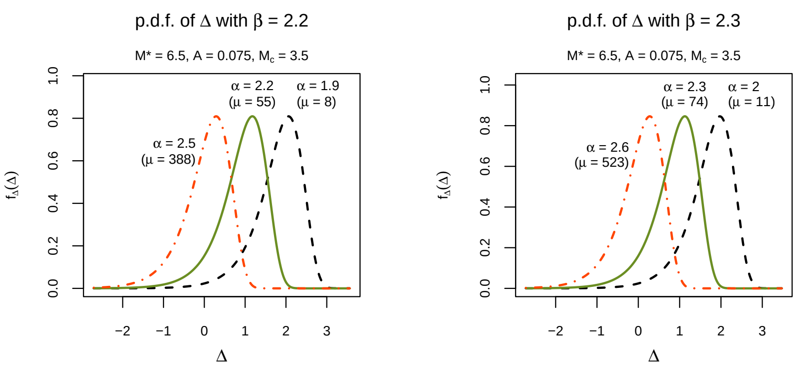

Figure 1: p.d.f. of $\Delta = M^* - M_{max}$ with $n \sim POI(Ae^{\alpha M^*})$. (“$\mu$” $= \mu(M_c \mid M^*)$)

Discussion and Conclusions

The simple model described here can be compared with those given by Console et al. (2003) and Shcherbakov (2005). The difference in appearance between the distributions described here and those of Console et al. may be attributable to the smoothing effect introduced by conditioning on M^* and allowing an averaging over $N(M^*)$. The equations can also be considered a reinterpretation of the ideas put forward in the papers by Shcherbakov et al., with the emphasis shifted to variations in the distribution of $N(M^*)$ as explaining the behaviour of Δ rather than vice-versa.

Although the derivations are straightforward, and the tools are clearly well-known to the writers cited earlier, as far as we can ascertain, and somewhat surprisingly, they have not previously been presented as a unified model. In fact the model supplies a general basis for anticipating a phenomenon closely similar to Båth's law without the need for a special physical mechanism, and at the same time gives a simple, general formula which can be used for estimating the probability that a given event will be followed by larger aftershocks. We are currently examining the extent to which it is supported by real data, and checking for any insight the parameters A, M_0, β and δ may provide into generating mechanisms.

References

Console, R., Lombardi, A.M. and Murru, M. (2003). Båth's law and the self-similarity of earthquakes. Journal of Geophysical Research, **108** N0 B2, 2128,doi: 1029/2001JB001651,2003.

Felzer, K.R., Abercrombie, R.E., and Ekström, G. (2004) A common origin for aftershocks, foreshocks and multiplets. Bulletin of American Seismological SAociety **94** 88-98

Helmstetter, A, and Sornette, D. (2003) Båth's law derived from the Gutenberg-Richter law and from aftershock properties. Preprint.

Lombardi, A.M. (2002) Probabilistic interpretation of Båth's law. Ann. Geophys. **45** 455-472.

Saichev, A. and Sornette, D. (2005) Distribution of the largest aftershocks in branching models of triggered seismicity: theory of Båth's law. (APS preprint)

Shcherbakov, R. and Turcotte, D.L. (2004) A modified form of Båth's law. Bulletin of American Seismological Society **94** 1968-1975

Shcherbakov, R. Turcotte, D.L., and Rundle, J.B. (2004) Aftershock statistics. Pure and Applied Geophysics **162** 1051-1076.

Zhuang, J. and Ogata, Y. (2005) Properties of the probability distributions associated with the largest event in an earthquake cluster and their implications to foreshocks. (preprint)